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Inaugural Lecture

What Quantitative Methods does a Macroeconomist Need to Know?

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1 Macroeconomic Models: Basic Issues

1.1 Multiplicity of Equilibria

- A linear univariate model

  $$y_t = \alpha + \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + v_t. \quad (1)$$

  where v_t is assumed to be an exogenous process satisfying $E_{t-1} v_t = 0$. $E_{t-1} y_t$ denotes expectation of y_t based on information at $t-1$ (like past values of y_t and v_t).

- Sargent-Wallace model (1975) and Taylor (1977) real balance model.

- (1) has large multiplicity of solutions.

- MSV (minimum state variable) solution. Use method of undetermined coefficients. Guess a solution of form

$$y_t = a + v_t,$$

i.e. a constant plus a shock where the constant a needs to be determined under rational expectations (RE). Assuming the guess to be correct

$$E_{t-1}y_t = E_{t-1}y_{t+1} = a.$$

- Substituting into equation (1), we get

$$y_t = \alpha + (\beta_0 + \beta_1)a + v_t,$$

This is consistent with our guess if and only if

$$a = \alpha + (\beta_0 + \beta_1)a$$

which implies

$$a = (1 - \beta_0 - \beta_1)^{-1}\alpha$$

and yields the rational expectations equilibrium (REE)

$$y_t = (1 - \beta_0 - \beta_1)^{-1}\alpha + v_t.$$

MSV sol depends linearly on set of variables (v_t and the intercept here) and there exists no solution which depends linearly on a smaller set of vars.

- But this is not all! There exist other solutions.

- $$y_t = -{\beta_1}^{-1}\alpha + {\beta_1}^{-1}(1-\beta_0)y_{t-1} + v_t + c_1 v_{t-1} + d_1 \varepsilon_{t-1}, \quad (2)$$

  is an REE for any choice of c_1 and d_1. Verify by computing $E_{t-1}y_t, E_{t-1}y_{t+1}$ from above and plugging into (1) (or use method of undetermined coefficients).

- Here ε_t is an arbitrary martingale difference sequence (MDS) i.e. a sequence of random variables satisfying $E_{t-1}\varepsilon_t = 0$. ε_t could be some function of v_t with conditional mean of 0, or it could be an exogenous variable independent of v_t (in the latter case we often call ε_t a sunspot variable).

- (2) often referred to as ARMA(1,1) set of solutions (for v_t white noise). (2) represents the full set of solutions. Can obtain MSV solution by deleting common lag polynomials and choosing appropriate values of c_1 and d_1.

- Insisting on stationary solutions does not eliminate multplicity. If ${\beta_1}^{-1}(1-\beta_0)$ is less than one in absolute value, ARMA class of sols are stationary (ultimately depends on underlying economic model).

- Note that if $\beta_1 = 0$, then only MSV sol exists i.e. unique solution to (1). Existence of forward expectation is key to multiplicity of solutions.

- Note if (1) has a lag as well i.e. is of form

$$y_t = \alpha + \delta y_{t-1} + \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + v_t. \tag{3}$$

then MSV solutions are of AR(1) form

$$y_t = \bar{a} + \bar{b} y_{t-1} + v_t$$

Compute $E_{t-1}y_t, E_{t-1}y_{t+1}$ and find that solution for $\bar{b}$ satisfies a quadratic equation so generically there exist two (or no) solutions. Even MSV sol is no longer unique.

- Also exist a continuum of ARMA(2,1) sunspot solutions. These are of form

$$\begin{aligned} y_t &= -\beta_1{}^{-1}\alpha + \beta_1{}^{-1}(1-\beta_0)y_{t-1} - \beta_1{}^{-1}\delta y_{t-2} \\ &\quad + v_t + c_1 v_{t-1} + d_1 \varepsilon_{t-1}, \end{aligned}$$

ε_t is a MDS. Even when no AR(1) sol exists, these ARMA sols are well defined.

- With two forward leads i.e. say

  $y_t = \alpha + \delta y_{t-1} + \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + \beta_2 E_{t-1} y_{t+2} +$

  where v_t white noise. MSV sols to this model are of AR(1) form

  $$y_t = \bar{a} + \bar{b} y_{t-1} + v_t$$

  with $\bar{b}$ given by solution to cubic equation so there are 3 or 1 such sol. In addition, there exist ARMA(3,2) and ARMA(2,1) class of solutions.

- Simple forward-looking scalar model

  $$y_t = A + M E_t y_{t+1} + \varepsilon_t,$$

  where ε_t is iid. A stationary MSV solution is

  $$y_t = (1 - M)^{-1} A + \varepsilon_t.$$

  In general define $y_{t+1} = E_t y_{t+1} + \mu_{t+1}$, where μ_{t+1} is an innovation with $E_t \mu_{t+1} = 0$ (MDS). Substituting in we obtain

  $$y_{t+1} = M^{-1} y_t - M^{-1} A - M^{-1} \varepsilon_t + \mu_{t+1}.$$

If $|M| < 1$, these processes are non-stationary. If $|M| > 1$, these processes are stationary for any μ_t so that there are multiple stationary solutions.

- Existence of (locally) unique stationary solution depends on the structure of the model.

Definition: If the system possesses a unique stationary REE, the model is said to be **determinate**. Otherwise the model is **indeterminate** in which case multiple stationary solutions exist or there may not exist any stationary REE. (The terminology "regular" and "irregular" models is also in use.)

- Multivariate models:
  Definition: a variable is **predetermined** if its value does not depend on expectations of the future (eg. capital in Ramsey model). Otherwise the variable is **free** or **non-predetermined (**eg. consumption in Ramsey model).

- For a multivariate model

$$\begin{aligned} y_t &= A + ME_t y_{t+1} + N y_{t-1} + P v_t, \\ v_t &= F v_{t-1} + \tilde{v}_t, \end{aligned}$$

group variables into free and predetermined variables x_t^1 and x_t^2, and write the model in the form

$$\begin{aligned} x_t^1 &= B_1 E_t x_{t+1}^1 + C x_t^2 \\ x_t^2 &= R x_{t-1}^1 + S x_{t-1}^2 + u_t. \end{aligned}$$

- **Blanchard-Kahn method** for computing a stationary solution:
Let $\eta_{t+1} = x_{t+1}^1 - E_t x_{t+1}^1$ and rewrite the system

$$\begin{aligned} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} &= J \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} + L \begin{pmatrix} u_{t+1} \\ \eta_{t+1} \end{pmatrix}, \text{ where} \\ J &= \begin{pmatrix} I & -C \\ R & S \end{pmatrix}^{-1} \begin{pmatrix} B_1 & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

Next, diagonalize J as

$$\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} J = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix},$$

where the number of eigenvalues in Λ_1 equals the number of free variables. (If J does not diagonalize, use the Jordan form.)

- We apply the following test:
  Test: For determinacy the number of non-predetermined (or free) variables must be the same as the number of eigenvalues inside the unit circle. If there are more free variables than roots inside the unit circle, the model is indeterminate.

- Continuing, we have

$$\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \Bigg($$

  Next, let $p_t = Q^{11}x_t^1 + Q^{12}x_t^2$ so the first equations are

$$p_t = \Lambda_1 p_{t+1} + L_1 u_{t+1} + L_2 \eta_{t+1}.$$

For i'th equation we have (assuming $\lambda_i \neq 0$)

$$E_t p^i_{t+1} = \lambda_i^{-1} p^i_t,$$

which explodes unless $p^i_t = 0$. This yields the stationary solution

$$\begin{aligned} x^1_t &= -(Q^{11})^{-1} Q^{12} [R x^1_{t-1} + S x^2_{t-1} + u_t] \\ x^2_t &= R x^1_{t-1} + S x^2_{t-1} + u_t. \end{aligned}$$

- **The irregular case** arises when the number of free variables is higher than the number of eigenvalues of J inside the unit circle.
  In this case there are multiple stationary RE solutions to the model.

Some of the solutions can depend on extraneous variables, called sunspots and there can exist **sunspot equilibria**.

2 Learning: Basic Ideas

- Rational expectations (RE) presupposes that economic agents know the structure of the economy. They do not know the unforecastable shocks.

- In practice we economists know much less and cannot form RE. How did the agents come to have RE?

- Alternative hypothesis: economic agents are like economists and use standard statistical techniques in forecasting.

- Learning can provide answers to several questions:
  (i) Can RE be attained if agents know much less?
  (ii) Which REE is attained when there are multiple REE?
  (iii)The dynamics under learning as an explanation of actual data.

3 Introduction

Consider the Lucas model:

$$\text{AS: } y_t^s = c_0 + c\,(p_t - p_t^e) + u_t,\ c > 0$$

$$\text{AD: } y_t^d = d_0 + d\,(m_t - p_t) + v_t,\ d > 0$$

For shocks we assume $E_{t-1}(u_t) = E_{t-1}(v_t) = 0$ and $\text{cov}(u_t, v_t) = 0$.

In equilibrium $AS = AD$:

$$p_t = \frac{d_0 - c_0 + dm_t + cp_t^e + v_t - u_t}{c + d}.$$

This is the solution for given expectations. To have a REE we need to make an assumption about price expectations and specify a policy rule.

Assume that $m_t = m$, a constant and that $\eta_t = (c+d)^{-1}(v_t - u_t)$ is iid. We rewrite the reduced form

$$p_t = \tilde{\mu} + \alpha p_t^e + \eta_t$$

where $\tilde{\mu} = (d_0 - c_0 + dm)/(c + d)$ and $\alpha = c/(c + d)$.

- Assuming RE the agents can compute

$$\begin{aligned} E_{t-1}p_t &= \tilde{\mu} + \alpha E_{t-1}p_t \implies \\ E_{t-1}p_t &= (1-\alpha)^{-1}\tilde{\mu} \text{ and} \\ p_t &= (1-\alpha)^{-1}\tilde{\mu} + \eta_t. \end{aligned}$$

- This computation presupposes agents know $\tilde{\mu}$ and α.

- What if agents do not know $\tilde{\mu}$ and α? Assume instead that agents have beliefs that the random process of p_t is of the form constant plus noise.

- The **perceived law of motion** (PLM) is then

$$p_t = a + \eta_t$$

and agents estimate a by a statistical method from data $\{p_i\}_{i=0}^{t-1}$.

- Let a_{t-1} be the estimate of a at the end of period $t-1$. The forecast for price is

$$p_t^e = a_{t-1}$$

and the temporary equilibrium given this forecast is

$$p_t = \tilde{\mu} + \alpha a_{t-1} + \eta_t,$$

which is a new data point that is used to get a new estimate a_t at the end of period t.

- A natural estimate of an unknown constant is the **sample mean** from the data:

$$a_{t-1} = t^{-1} \sum_{i=0}^{t-1} p_i.$$

What happens as $t \to \infty$?

Result: $a_t \to (1-\alpha)^{-1}\tilde{\mu}$ if $\alpha < 1$ and a_t does not converge to $(1-\alpha)^{-1}\tilde{\mu}$ if $\alpha > 1$.

Note: In the example $\alpha = c/(c+d) < 1$, so we have convergence.

3.1 An Exogenous Observable

More generally, assume that m_t follows the stochastic process

$$m_t = w_{t-1} + u_t$$

where w_{t-1} is a series of observable (demand/supply) shocks and u_t is white noise. We rewrite the reduced form

$$p_t = \mu + \alpha p_t^e + \delta w_{t-1} + \eta_t$$

where now $\mu = (d_0 - c_0)/(c+d)$ and $\delta = d/(c+d)$. Under RE $p_t^e = E_{t-1}p_t$, and we obtain

$$E_{t-1}p_t = (1-\alpha)^{-1}\mu + (1-\alpha)^{-1}\delta w_{t-1}$$

There is a unique REE

$$p_t = (1-\alpha)^{-1}\mu + (1-\alpha)^{-1}\delta w_{t-1} + \eta_t.$$

If agents do not know μ, α and δ they might believe that prices follow

$$p_t = a + b w_{t-1} + \eta_t, \qquad (4)$$

but that a and b are unknown to them. (4) is the PLM and the agents attempt to estimate a and b using data up to $t-1$:

(i) Agents have data on the economy from periods $i = 0, ..., t-1$, the time $t-1$ information set is $\{p_i, w_i\}_{i=0}^{t-1}$ at the time of estimation.

(ii) Agents estimate a and b by a least squares regression of p_i on w_{i-1} and an intercept and make forecasts using the estimated model.

(iii) Their estimates will be updated over time as more information is collected.

Letting (a_{t-1}, b_{t-1}) denote the estimates through time $t-1$, their forecasts at $t-1$ are given by

$$p_t^e = a_{t-1} + b_{t-1} w_{t-1} \tag{5}$$

and the actual (or temporary equilibrium) price in period t is

$$p_t = \mu + \alpha(a_{t-1} + b_{t-1} w_{t-1}) + \delta w_{t-1} + \eta_t$$

The standard least squares formula gives the equations

$$\begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} = \left(\sum_{i=1}^{t-1} z_{i-1} z'_{i-1}\right)^{-1} \left(\sum_{i=1}^{t-1} z_{i-1} p_i\right),$$

$$\text{where } z'_i = \begin{pmatrix} 1 & w'_i \end{pmatrix}. \qquad (6)$$

The question of interest is whether $a_t \to \bar{a} \equiv (1 - \alpha)^{-1}\mu$ and $b_t \to \bar{b} \equiv (1 - \alpha)^{-1}\delta$ as $t \to \infty$. Result:

Theorem 3.1 *If* $\alpha < 1$ *then* $\begin{pmatrix} a_t \\ b_t \end{pmatrix} \to \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}$ *with probability 1. If* $\alpha > 1$ *then convergence occurs with probability 0.*

3.2 Expectational Stability

The condition $\alpha < 1$ follows from a general stability principle, "Expectational Stability" or "E-stability".

The E-stability principle: the mapping from the PLM to the **actual law of motion** (ALM) governs the stability of equilibria under least squares learning.

(i) Agents are assumed to have a PLM which they use to make forecasts. Usually the functional form of the PLM corresponds to the REE of interest. For the natural rate model the PLM is

$$p_t = a + bw_{t-1} + \eta_t.$$

(ii) For any given a and b the time $t-1$ forecast of p_t is

$$p_t^e = a + bw_{t-1}. \tag{7}$$

(iii) One solve for the ALM:

$$p_t = (\mu + \alpha a) + (\delta + \alpha b)w_{t-1} + \eta_t. \tag{8}$$

This implicitly defines the **mapping from the PLM to the ALM**

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix}. \tag{9}$$

Interpretation: the ALM describes the stochastic process followed by the economy if forecasts are made under the fixed PLM.

(iv) E-stability is defined using $T-$map and the differential equation

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}, \tag{10}$$

where τ denotes “notional” or “artificial” time. Note that the REE is a fixed point of the $T-$map.

The REE is **expectationally stable**, or **E-stable**, if the REE is locally asymptotically stable under (10).

In the model

$$\begin{aligned} \frac{da}{d\tau} &= \mu + (\alpha - 1)a \\ \frac{db}{d\tau} &= \delta + (\alpha - 1)b. \end{aligned}$$

It follows that the REE is E-stable if and only if $\alpha < 1$.

4 Recursive Least Squares

Why does the E-stability analysis give convergence conditions for least squares learning?

We return to adaptive learning and recall that agents are running a least squares regression of p_i on z_{i-1}, where $z_i' = (\ 1 \quad w_i\ )$. For convenience, write

$$\phi_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix}.$$

Write least squares in recursive form, i.e. recursive least squares (RLS):

$$\begin{aligned} \phi_t &= \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1}(p_t - \phi_{t-1}' z_{t-1}) \\ R_t &= R_{t-1} + t^{-1}(z_{t-1} z_{t-1}' - R_{t-1}). \end{aligned}$$

For p_t we have

$$p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1}) w_{t-1} + \eta_t, \text{ or}$$

$$p_t = T(\phi_{t-1})' z_{t-1} + \eta_t. \qquad (11)$$

where $T(\phi)$ is as above. We arrive at the stochastic recursive system

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1}(z_{t-1}'(T(\phi_{t-1}) - \phi_{t-1}) + \eta_t) \qquad (12)$$

$$R_t = R_{t-1} + t^{-1}(z_{t-1} z_{t-1}' - R_{t-1}). \qquad (13)$$

We want to show that under (12)-(13) $\phi_t \to \bar{\phi}$ with probability 1 as $t \to \infty$ when $\alpha < 1$. Note that also the price process converges to the REE.

4.1 Convergence of Stochastic Recursive Algorithms

We consider a **stochastic recursive algorithm** (SRA) of the form

$$\theta_t = \theta_{t-1} + \gamma_t Q(t, \theta_{t-1}, X_t), \tag{14}$$

where θ_t is a vector of parameter estimates, X_t is the state vector and γ_t is a deterministic sequence of "gains".

In our example, θ_{t-1} includes ϕ_{t-1} and R_t, X_t includes the effects of z_{t-1} and η_t, and $\gamma_t = t^{-1}$.The stochastic approximation approach associates an ordinary differential equation with the SRA,

$$\frac{d\theta}{d\tau} = h(\theta(\tau)),$$

where $h(\theta)$ is obtained as

$$h(\theta) = \lim_{t\to\infty} EQ(t, \theta, X_t), \tag{15}$$

provided this limit exists. E denotes the expectation of $Q(t, \theta, X_t)$, for θ fixed, taken over the invariant distribution of X_t.
Note: If X_t depends on θ_{t-1}, then one needs to use

$$h(\theta) = \lim_{t \to \infty} EQ(t, \theta, \bar{X}_t(\theta)),$$

where $\bar{X}_t(\theta)$ is the stochastic process for X_t with θ_{t-1} held at the fixed value $\theta_{t-1} = \theta$.

The stochastic approximation results can be stated as follows:

Under suitable assumptions, if $\bar{\theta}$ is a locally stable equilibrium point of the ODE then $\bar{\theta}$ is a possible point of convergence of the SRA. If $\bar{\theta}$ is not a locally stable equilibrium point of the ODE then $\bar{\theta}$ is not a possible point of convergence of the SRA, i.e. $\theta_t \to \bar{\theta}$ with probability 0.

Remarks:
(A) Care must be taken with the sense of convergence

$\theta_t \to \bar{\theta}$.

(B) Certain technical assumptions must be met:

(i) Regularity assumptions on Q.

(ii) Conditions on the rate at which $\gamma_t \to 0$, often $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 < \infty$. (Example: $\gamma_t = t^{-1}$.)

(iii) Assumptions on the stochastic process followed by X_t.

4.2 Application to the model

We need a timing change $S_{t-1} = R_t$. The system (12)-(13) becomes

$$\phi_t = \phi_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} (z_{t-1}' (T(\phi_{t-1}) - \phi_{t-1}) + \eta_t) \tag{16}$$

$$S_t = S_{t-1} + t^{-1} \left(\frac{t}{t+1}\right) (z_t z_t' - S_{t-1}). \tag{17}$$

This system is now in standard form with:

$$\theta_t = \text{vec} \begin{pmatrix} \phi_t & S_t \end{pmatrix}, X_t' = \begin{pmatrix} 1 & w_t & w_{t-1} & \eta_t \end{pmatrix}, \gamma_t = t^{-1}.$$

Here vec stacks in order the columns of the matrix $\begin{pmatrix} \phi_t & S_t \end{pmatrix}$ into a column vector. The function $Q(t, \theta_{t-1}, X_t)$ in the SRA is now fully specified.

The associated ODE can be shown to be

$$\frac{d\phi}{d\tau} = S^{-1} M (T(\phi) - \phi) \quad (18)$$

$$\frac{dS}{d\tau} = M - S, \quad (19)$$

where

$$E z_t z_t' = M.$$

This system is recursive and the second set of equations is a globally stable system with $S \to M$. Hence that the stability of (18)-(19) is determined entirely by the stability of

$$\frac{d\phi}{d\tau} = T(\phi) - \phi. \quad (20)$$

Recalling that $\phi' = \begin{pmatrix} a & b \end{pmatrix}$, (20) is identical to (10) which defines E-stability. $\bar{\phi}' \equiv (\bar{a}, \bar{b})$ is stable provided $\alpha < 1$.

4.3 New Keynesian model

- The model

$$x_t = -\varphi[i_t - E_t\pi_{t+1}] + E_t^* x_{t+1} + g_t, \quad (21)$$
$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t \quad (22)$$

x_t is the output gap, π_t is the inflation rate, and i_t is the nominal interest rate; all variables are percentage deviations from their steady-state values. $E_t\pi_{t+1}$ and $E_t x_{t+1}$ denote private sector expectations of inflation and output gap. φ, λ are positive & $0 < \beta < 1$ is the discount rate of the representative firm.

- IS curve relates the "output gap" (the difference between actual output and the natural level of output i.e. the equilibrium level of output with flexible prices and wages) inversely to the real rate of interest. Phillips curve relates inflation positively to the output gap.

- Shock g_t as a demand or government spending shock. u_t represents cost push shocks to marginal costs.

- (21) is a dynamic "IS" curve that can be derived from the inter-temporal first order condition for consumption (the Euler equation) associated with the household's optimal savings decision after imposing the equilibrium condition that consumption equals output minus government spending.

- (22) is a "new Phillips curve" that can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. Monetary policy is conducted by the central bank by means of control of the nominal interest rate i_t.

- This model is complete when a rule for monetary policy is specified. This is normally specified by a rule to set the interest rate.

- **Instrument rules** for i_t are without consideration of policy optimization. A prominent example is the **Taylor** (1993) rule

$$i_t = \varpi + \chi_\pi(\pi_t - \bar{\pi}) + \chi_x(x_t - \bar{x}), \qquad (23)$$

  where $\bar{\pi}$ is the target level of inflation and the target level of the output gap is $\bar{x}$. Sometimes non-policy shocks are also included. Taylor used the numbers $\chi_\pi = 1.5$ and $\chi_x = 0.5$.

- More generally Taylor-type rules are, wlog setting $\bar{\pi} = 0$,

$$i_t = \chi_\pi \pi_t + \chi_x x_t, \text{ where } \chi_\pi, \chi_x > 0 \qquad (24)$$

- Variations of the Taylor rule replace π_t and x_t by lagged values

$$i_t = \chi_\pi \pi_{t-1} + \chi_x x_{t-1}$$

  or by forecasts of current or future values like

$$i_t = \chi_\pi E_t \pi_{t+1} + \chi_x E_t x_{t+1}$$

or

$$i_t = \chi_\pi E_{t-1}\pi_t + \chi_x E_{t-1}x_t$$

4.3.1 Determinacy Result for Taylor rule

- Consider the Taylor-type rule where

  $$i_t = \chi_\pi \pi_t + \chi_x x_t, \text{ where } \chi_\pi, \chi_x > 0.$$

  Plug this interest rule in the structural equations (21) and (22) to obtain the following system (note there are no lagged endogenous variables with this rule)

  $$y_t = \alpha + M E_t^* y_{t+1} + \varkappa v_t \tag{25}$$

  where $y_t = [x_t, \pi_t]'$, $\alpha = 0$, and

  $$M = \frac{1}{\varphi^{-1} + \chi_x + \lambda\chi_\pi}\left[\begin{array}{cc} \varphi^{-1} & 1 - \beta\chi_\pi \\ \lambda\varphi^{-1} & \lambda + \beta\left(\varphi^{-1} + \chi_x\right) \end{array}\right], \tag{26}$$

  Since both x_t and π_t in the system (25) are free, we need both of the eigenvalues of M to be inside the unit circle for determinacy; otherwise the equilibrium will be indeterminate.

- **Note:** The eigenvalues of a 2×2 matrix M are inside the unit circle if and only if $|\text{det}(M)| < 1$ and $|tr(M)| < 1 + \text{det}(M)$. One can apply this result here and show that the basic Taylor rule yields determinacy iff

$$\chi_\pi + \chi_x \frac{(1-\beta)}{\lambda} > 1.$$

- Same condition determines E-stability!

Figure 1

Policy Rules with Contemporaneous Data

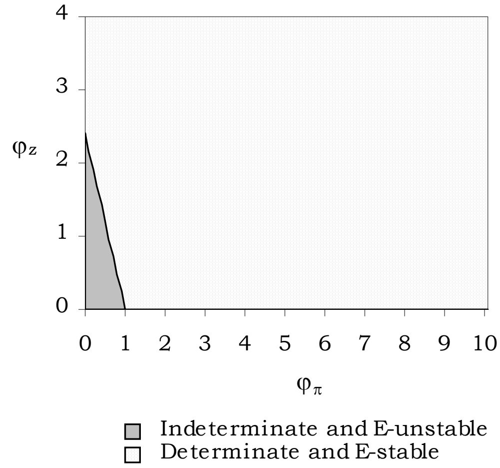

Figure 1: