Basic Problem

$$\max_{a \in A_s} F(s, a)$$

Question: when is $a^*(s) \nearrow$ in s?

Implicit theorem approach:

For an interior solution,

$$F_a(s, a(s)) = 0 \quad \rightarrow \quad C^1$$

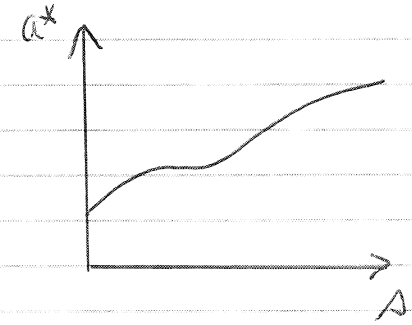
Assumptions: $F_{aa} \neq 0$, $F_{aa} < 0$ $\rightarrow$ C^2

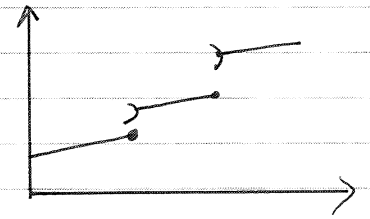
differentiate:

$$\Rightarrow F_{as}(s, a(s)) + F_{aa}\, a'(s) = 0 \rightarrow a'(s) = -\frac{F_{as}}{F_{aa}}\Big|_* \geq 0$$

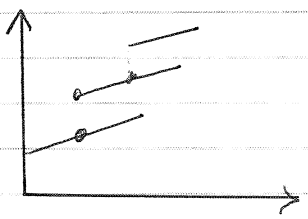
(F_{as}: $\oplus$; F_{aa}: $\ominus$)

$F_{as}^{\oplus}$ condition cannot be avoided





method cannot work



multiple solutions, etc

but we could take a selection

$F: \mathbb{R}^2 \to \mathbb{R}$

F has increasing diff (ID) if $\forall\ a' > a,\ s' > s$,

$$F(s', a') - F(s', a) \geq F(s, a') - F(s, a)$$

$$\Leftrightarrow \quad F'(s', a) - F(s, a) \nearrow \text{ in } a$$

<u>Lemma</u> If F is C^2, then F has ID $\Leftrightarrow F_{as} \geq 0,\ \forall a, s$

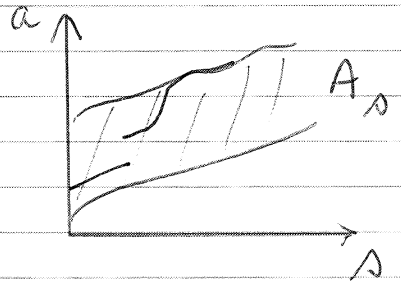
<u>Proof</u> F has ID $\Leftrightarrow \frac{\partial}{\partial a}\left[F(s', a) - F(s, a)\right] \geq 0$

$$\Leftrightarrow \frac{\partial}{\partial a} F(s', a) \geq \frac{\partial}{\partial a} F(s, a)$$

$$\Leftrightarrow \frac{\partial}{\partial a} F(s, a) \nearrow \text{ in } s$$

$$\Leftrightarrow \frac{\partial}{\partial s}\frac{\partial}{\partial a} F(s, a) \geq 0$$

Now we now need to have assumptions on A_s

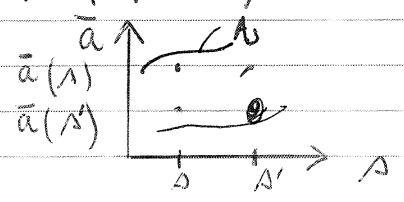


A_s is ascending if $A_s = [g(s), h(s)]$ with $g, h \nearrow$ with $g \leq h$

<u>Topkis's THM</u> : If F has ID and A_s is ascending, then

$\underline{a}(s) = \min a(s)$ and $\bar{a}(s) = \max a(s)$ are $\nearrow$ in s

<u>Proof</u> (for $\bar{a}$) Assume there is $s' > s$ s.t. $\bar{a}(s') < \bar{a}(s)$



Since F has ID, $F(s', \bar{a}(s)) - F(s', \bar{a}(s')) \geq$

$F(s, \bar{a}(s)) - F(s, \bar{a}(s')) \geq 0$

but $F(s', \bar{a}(s)) - F(s', \bar{a}(s')) \leq 0$

and $F(s', \bar{a}(s)) - F(s', \bar{a}(s')) = F(s, \bar{a}(s)) - F(s, \bar{a}(s'))$

$= 0$

$\bar{a}(s) \in a^*(s)$, a contradiction to the definition of $\vec{a}(\cdot)$.

Note We use A_s ascending in taking $F(s, \bar{a}(s'))$ and $F(s', \bar{a}(s))$

- Same for $\underline{a}$.
- Strict

Remark With interiority

Replace ID with ($F \in C^1$) $\frac{\partial F}{\partial a}(s, a)$ strictly ↑ in s

then $\bar{a}(s)$ is strictly ↑

Example 1 $\max U(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 \leq I$

Q: When is good 1 normal ($x_1^* \nearrow$ in I)

$x_2 = \frac{1}{p_2}(I - p_1 x_1)$

$\max_{0 \leq x_1 \leq \frac{I}{p_1}} U\left(x_1, \frac{1}{p_2}(I - p_1 x_1)\right)$

Check ID in (x_1, I)

Take ∂x_1: $U_1 - \frac{p_1}{p_2} U_2$

$\partial x_1 \partial I$: $\frac{1}{p_2} U_{12} - \frac{p_1}{p_2^2} U_{22} \overset{ID}{\geq} 0$

$\Longleftrightarrow$ $p_2 U_{12} - p_1 U_{22} \geq 0$

U needs to be:
smooth
quasi-concave
directs units OK

Example 2 : Monopoly

$$\max_{p \geq 0} \pi(p, c) = (p - c) D(p)$$

Q: When is $p^*(c)$ ↗ ?

We need π to have ID

$$\frac{\partial \pi}{\partial c} = -D(p)$$

$$\frac{\partial^2 \pi}{\partial c \partial p} = -D'(p) \geq 0 \quad \text{if } D' \leq 0$$

$$\log \pi(p, c) = \log (p - c) + \log D(p)$$

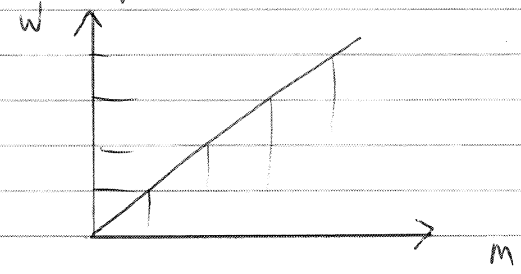
$$\frac{\partial^2 \log \pi}{\partial p \partial c} = \frac{1}{(p - c)^2} \geq 0 \rightarrow \text{Always}$$

→ log-concave

→ z^* ↓ in c

→ $\frac{\partial p^*}{\partial c} \leq 1$

Example 3



$\max \sum_{(i,j)} f(i,j)$ Assortative matching as unique sol'
$\Leftarrow$ f has ID

4) Cournot ~~Sta~~ duopoly

$$\Pi(x, y) = x\, P(x+y) - C(x)$$

The game has SS if $r(y) = \arg\max_{x \geq 0} \Pi(x, y)$ $\downarrow$ in y?

$$\frac{\partial \Pi}{\partial x} = P + x P' - C'$$

$$\frac{\partial^2 \Pi}{\partial x \partial y} = P'(x+y) + x P''(x+y) \leq 0 \quad , \forall (x, y)$$

Supermodular games

$$(F, G, X, Y)$$

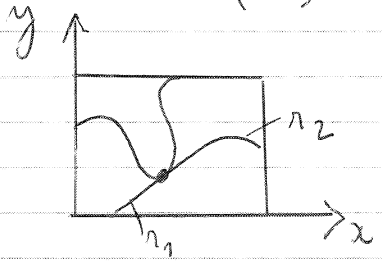
$X = [0, 1]$, $Y = [0, 2]$

$F : X \times Y \longrightarrow \mathbb{R}$
G

Def: A game is spm if F and G have ID

Outcome: $r_1(y)$ is in $\longrightarrow$ in y

$r_2(x)$ ——————— in x



B: $X \times Y \longrightarrow X \times Y$

$(x, y) \longrightarrow (r_1(y), r_2(x))$

$(\bar{x}, \bar{y}) = B(\bar{x}, \bar{y})$

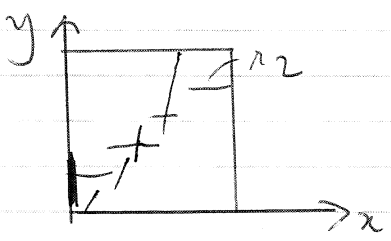
B continuous $\rightarrow$ Brouwer's

spm: r_1 and r_2 are in $\nearrow$ B is $\nearrow$

Tarski THM

$$T : \bigtimes_{i=1}^{n} [a_i, b_i] \longrightarrow \times [a_i, b_i]$$

T $\nearrow$ $\Rightarrow$ T has a fixed point



game:

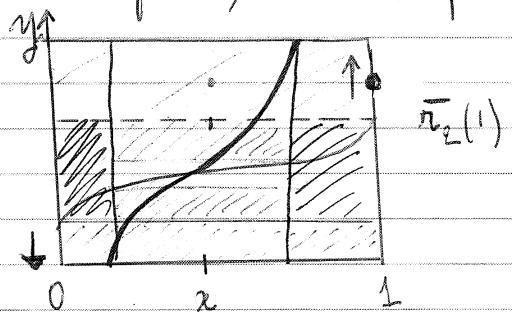
2,5	0,0	1,4
3,3	0,0	2,5
2,2	1,2	3,6

unique pure strategy Nash eq

Def U = set of strategies that survive $IESDS$

THM : $\underline{a} = \inf U$, $\bar{a} = \sup U$

Proof



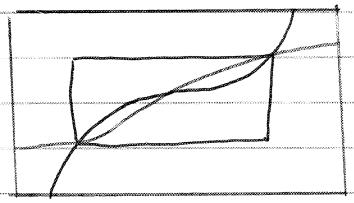
$\forall x \geq 0$

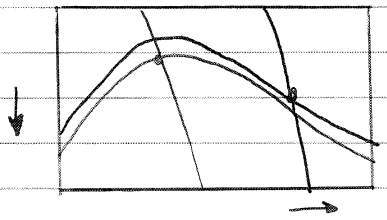
$G(x, r_2(1)) > G(x, y)$, $\forall y > r_2(1)$

$0 > G(1, y) - G(1, r_2(1)) \geq$

$G(x, y) - G(x, r_2(1))$

etc $\rightsquigarrow$ unique equilibrium

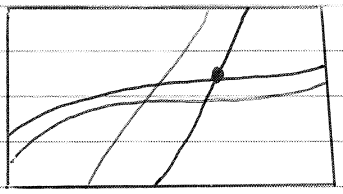




$c_1 + t$

$c_2 + t$

$r_2(y, t) = \arg\max_x F(x, y, t)$

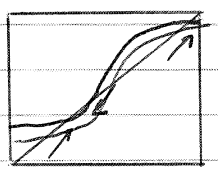


Strategic complement

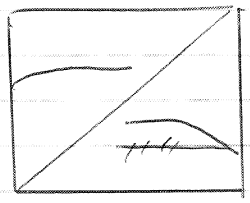
THM $F(x, y, t)$, $G(x, y, t)$ has $\nearrow$ equilibria.

- F and G have ID
- F has ID in (x, t)
  G has ID in (y, t)

Samuelson (40s)



sub modular games



$R = r_1 \circ r_2 (x) \longrightarrow x$

$\bar{x} = r_1 \circ r_2 (\bar{x})$ or $\bar{x} = r_1 [r_2 (\bar{x})]$

$(\bar{x}, r_2 (\bar{x}))$