Topics in Nonlinear Economic Dynamics:
Bounded Rationality, Heterogeneous Expectations
and Complex Adaptive Systems

Lecture 1b: A behavioral asset pricing model
with heterogeneous expectations

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Main ingredients of Lecture 1b:

- simple **behavioral** asset pricing model with heterogeneous expectations
- numerical **simulations**
- **empirical validation**: estimation of simple 2-type example
- afternoon: **laboratory experiments** with human subjects
Some Literature


Traditional Approach

- representative agent, who is perfectly rational
- **Friedman hypothesis**: “irrational agents will lose money and will be driven out the market by rational agents”
- simple (linear), stable model, driven by exogenous random news about fundamentals (no predictability, no arbitrage)
- prices reflect economic fundamentals (market efficiency)
Some Drawbacks Traditional Approach

- extreme, **unrealistic assumptions** about agents’ perfect knowledge and super computing abilities
- **no trade** results contradict huge trading volume
- **can not explain stylized facts**, e.g. excess volatility, bubbles and crashes, fat tails, clustered volatility etc.
Ken Arrow, 2004:

“One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn’t have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.”
Heterogeneous, interacting agents approach

- **heterogeneous** agents, heterogeneous beliefs
- market **psychology**, **herding** behavior (Keynes (1936))
- **bounded rationality** (Simon (1957))
- markets as **complex adaptive, nonlinear evolutionary** systems
- **interaction** of agents creates **aggregate structure** explaining stylized facts
Some Drawbacks Interacting Agents Approach

- ‘wilderness’ of bounded rationality
- many degrees of freedom for heterogeneity
- what exactly causes the outcome in a (large) computational HAM
General Framework HAMs: evolutionary strategy selection

Brock (93), Blume (93), Brock-Hommes (97,98), Brock-Durlauf (2001)

Survival of the “fittest”: strategies that have performed well in the recent past attract more followers; strategy updating according to fitness measure

\[ U_{ht} = \pi_{ht} + S_{ht} + \varepsilon_{ht}, \]

- \( \pi_{ht} \): private utility, e.g. accumulated profit, forecasting performance
- \( S_{ht} \): social utility, e.g. herding, mean opinion index, social interactions
- \( \varepsilon_{ht} \): noise, random utility
Example: discrete choice/multinomial logit model

As the number of agents $N \to \infty$, the probability of choosing strategy $h$ tends to the deterministic fraction

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{Z_{t-1}},$$

where $Z_{t-1} = \sum \exp(\beta U_{h,t-1})$ is normalization factor, $\beta$ is the intensity of choice

- $\beta = 0$: random choice, all fractions equal weight
- $\beta = \infty$: ‘neoclassical limit’ all agents choose “best” strategy
Behavioral asset pricing model (Adaptive Belief Systems)
(Brock and Hommes (Econometrica 1997, JEDC 1998))

standard asset pricing model with heterogeneous beliefs
one risky asset, one risk free asset

- price of risky asset determined by market clearing
- beliefs about future prices given by simple, linear rule
- forecasting strategies updated according to discrete choice model with realized profits

\[ R = 1 + r > 1: \text{ gross return on risk free asset} \]

\[ p_t: \text{ price (ex div.) per share of risky asset} \]
\[ y_t: \text{ stochastic dividend process for risky asset} \]
\[ z_{ht}: \text{ demand for risky shares by agents of type } h \]
\[ n_{ht}: \text{ fraction of agents of type } h \]
Asset pricing model with heterogeneous beliefs (Ctd)

Myopic mean-variance maximization of expected wealth demand for risky shares by type $h$:

$$z_{ht} = \frac{E_{ht}(p_{t+1} + y_{t+1} - R_p)}{aV_{ht}(p_{t+1} + y_{t+1} - R_p)} = \frac{E_{ht}(p_{t+1} + y_{t+1} - R_p)}{a\sigma^2},$$

(conditional variance assumed to be constant, i.e. $V_{ht} \equiv \sigma^2$, for all types and $a$ is risk aversion parameter)

Market clearing

$$\sum_{h=1}^{H} n_{ht} \frac{E_{ht}(p_{t+1} + y_{t+1} - R_p)}{a\sigma^2} = z^s$$
Asset pricing model with heterogeneous beliefs (Ctd)

**Equilibrium pricing equation**

\[ Rp_t = \sum_{h=1}^{H} n_{ht} E_{ht}(p_{t+1} + y_{t+1}) - a\sigma^2 z^s \]

**special case:** constant zero supply of outside shares \( z^s_t = 0 \)

\[ Rp_t = \sum_{h=1}^{H} n_{ht} E_{ht}(p_{t+1} + y_{t+1}) \]

**Remark:** this corresponds to risk-neutral agents
Homogeneous beliefs: RE fundamental benchmark

\[ Rp_t = E_t(p_{t+1} + y_{t+1}) \]

**common beliefs** on future earnings and prices
unique bounded RE **fundamental price** \( p_t^* \):

\[ p_t^* = \frac{E_t(y_{t+1})}{R} + \frac{E_t(y_{t+2})}{R^2} + \ldots \]

For special case of IID dividends, with \( E(y_{t+1}) = \bar{y} \)

\[ p_t^* = \frac{\bar{y}}{R - 1} = \frac{\bar{y}}{r} \]

pricing equation in **deviations** \( x_t = p_t - p_t^* \) from fundamental:

\[ Rx_t = E_t x_{t+1} \]

**Note:** rational bubbles growing at rate \( R = 1 + r \).
Heterogeneous Beliefs

Assumptions about beliefs of trader type $h$:
special case: IID dividends, i.e. $E[y_{t+1}] = \bar{y}$

**B1** $V_{ht}[p_{t+1} + y_{t+1} - Rp_t] = V_t[p_{t+1} + y_{t+1} - Rp_t] = \sigma^2$.

**B2** $E_{ht}[y_{t+1}] = E_t[y_{t+1}] = E[y_{t+1}] = \bar{y}$.

**B3** All beliefs $E_{ht}[\tilde{p}_{t+1}]$ are of the form

$$E_{ht}[p_{t+1}] = E_t[p^*_t] + E_{ht}[x_{t+1}] = E_t[p^*_t] + f_h(x_{t-1}, \ldots, x_{t-L})$$

$$= p^*_{t+1} + f_h(x_{t-1}, \ldots, x_{t-L}) = p^* + f_h(x_{t-1}, \ldots, x_{t-L}).$$

Under **B1-B3** the **equilibrium pricing equation** can be rewritten in deviations $x_t = p_t - p^*_t$ from the fundamental:

$$Rx_t = \sum_{h=1}^{H} n_{ht} E_{ht} x_{t+1} = \sum_{h=1}^{H} n_{ht} f_{ht}$$
Examples of simple forecasting rules

In few type examples, typical simple forecasting rules have been:

- **linear forecasting rule** as in BH 1998:
  \[ E_{ht}(x_{t+1}) = f_{ht} = gx_{t-1} + b \]

- **fundamentalists**: prices move towards fundamental price
  \[ E_{ht}(p_{t+1}) = p^* + \nu(p_{t-1} - p^*), \quad 0 \leq \nu < 1 \]

- **trend extrapolator** as in GH 2000:
  \[ E_{ht}(p_{t+1}) = p_{t-1} + g(p_{t-1} - p_{t-2}), \quad g > 0. \]
Evolutionary Fitness Measure

realized excess return: \( R_t = p_t + y_t - Rp_{t-1} \)
demand by type \( h \):

\[
z_{h,t-1} = \frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{a\sigma^2}
\]

realized net profits in period \( t + 1 \)

\[
\pi_{ht} = R_t z_{h,t-1} = (p_t + y_t - Rp_{t-1}) \frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{a\sigma^2} = (x_t - Rx_{t-1}) \frac{E_{h,t-1}[x_t - Rx_{t-1}]}{a\sigma^2}
\]

Fitness function or performance measure
weighted sum of realized profits; \( w \) memory strength

\[
U_{ht} = \pi_{ht} + wU_{h,t-1}
\]
Fractions of Strategy Type $h$

fractions of belief types are updated in each period: discrete choice model (BH 1997,1998) with asynchronous updating:

$$n_{ht} = (1 - \delta) \frac{e^{\beta U_{h,t-1}}}{Z_{t-1}} + \delta n_{h,t-1},$$

where $Z_{t-1} = \sum e^{\beta U_{h,t-1}}$ is normalization factor, $U_{h,t-1}$ past strategy performance, e.g. (weighted average) past profits.

$\delta$ is probability of not updating
$\beta$ is the intensity of choice.
$\beta = 0$: all types equal weight (in long run)
$\beta = \infty$: fraction $1 - \delta$ switches to best predictor
Example with 4 belief types

(zero costs; memory one lag)

\[ x_{h,t+1}^e = g_h x_{t-1} + b_h \]

\[
\begin{align*}
  g_1 &= 0 & b_1 &= 0 & \text{fundamentalists} \\
  g_2 &= 1.1 & b_2 &= 0.2 & \text{trend + upward bias} \\
  g_3 &= 0.9 & b_3 &= -0.2 & \text{trend + downward bias} \\
  g_4 &= 1.21 & b_4 &= 0 & \text{trend chaser}
\end{align*}
\]

\[
R_{x_t} = \sum_{h=1}^{4} n_{h,t} (g_h x_{t-1} + b_h)
\]

\[
n_{h,t+1} = \frac{e^{\beta \frac{1}{\sigma^2} (g_{h,x_{t-2}+b_h-R_{x_{t-1}}})(x_{t}-R_{x_{t-1}})}}{Z_t}, \quad h = 1, 2, 3, 4
\]
Route to complicated dynamics

- $\beta < \beta^*$: fundamental steady state **globally stable**
- $\beta = \beta^*$: **Hopf bifurcation** of steady state
- $\beta^* < \beta < \beta^{**}$: (quasi-)**periodic** price fluctuations on attracting invariant circle
- high values of $\beta$: **strange attractors**
- $\beta = \infty$: convergence to (locally unstable) fundamental steady state
Rational Route to Randomness
(complicated dynamics for large intensity of choice)
Strange Attractor (4 belief types)
Sensitivity to Noise (4 belief types)
NNB Forecasting Performance with and without Noise

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Conclusions

- **larger intensity of choice** (i.e. sensitivity to differences in fitness) leads to *instability*;
  - random choice $\rightarrow$ stability
  - correlated choice $\rightarrow$ instability
- fundamentalists **do not** drive out chartists, according to realized (short run) profits
- **market inefficiency**: asset prices *deviate persistently* from benchmark fundamental
- **close to efficiency**: asset prices are irregular and **difficult to predict**
Empirical validation:

- how relevant are these **bubble and crash dynamics** to real financial data?
- **estimation** of simple heterogeneous agent model with **two types** and **evolutionary switching** on yearly S&P 500 data
S&P 500, 1871-2003 + benchmark fundamental \( p_t^* = \frac{1+g}{1+r} y_t \)

(g constant growth rate dividends)

S&P 500 and log fundamental

PD-ratio and PD-fundamental
Two competing explanations in asset valuation

From Fama and French (2002):

“... there are two schools of thought on how to explain the variation in expected returns. Some attribute it to rational variation in response to macroeconomic factors ... while others judge that irrational swings in investor sentiment are the prime moving force.”
fundamental benchmark for constant growth cash flow

(Gordon model)

cash flow with growth rate $g$: $y_{t+1} = (1 + g)y_t \varepsilon_{t+1}, \quad \varepsilon_t \sim N(1, \sigma^2_\varepsilon)$

$$p_t = \frac{1}{1+r} E_t(p_{t+1} + y_{t+1})$$

unique bounded RE fundamental price $p_t^*$:

$$p_t^* = \frac{E_t(y_{t+1})}{1+r} + \frac{E_t(y_{t+2})}{(1+r)^2} + \ldots = \frac{1 + g}{1+r} y_t + \frac{(1 + g)^2}{(1+r)^2} y_t + \ldots = \frac{1 + r}{r - g} y_t.$$  

fundamental price to cash flow ratio $\delta_t^* = \frac{p_t^*}{y_t} = \frac{1+r}{r-g} = m$
Heterogeneous beliefs model in terms of price to cash flows

declaration price-to-cash flow from fundamental

\[ x_t = \delta_t - m = \delta_t - \frac{1+g}{r-g} \]

belief of type h about price-to-cash flow:

\[ E_{ht}[\delta_{t+1}] = E_t[\delta^*_t] + f_h(x_{t-1}, \ldots, x_{t-L}) = m + f_h(x_{t-1}, \ldots, x_{t-L}) \]

estimation of two type model (in deviations from fundamental)

\[ R^* x_t = n_t \phi_1 x_{t-1} + (1 - n_t) \phi_2 x_{t-1} + \varepsilon_t \]

\[ R^* = \frac{1+r}{1+g} \]
Estimation for the PD ratio

\[ R^* x_t = n_t \{ 0.762 x_{t-1} \} + (1 - n_t) \{ 1.135 x_{t-1} \} + \hat{\epsilon}_t \]

(0.056)  (0.036)  

(1)

\[ n_t = \left\{ 1 + \exp\left[ -10.29(-0.373 x_{t-3})(x_{t-1} - R^* x_{t-2}) \right] \right\}^{-1} \]

(6.94)

(2)

\[ R^2 = 0.82, AIC = 3.18, AIC_{AR(1)} = 3.24, \sigma\epsilon = 4.77, Q_{LB}(4) = 0.44 \]

fraction of type 1 coefficient \( \phi_t = \{ n_t \phi_1 + (1 - n_t) \phi_2 \} / R^* \)
Will the bubble resume?

Quantiles of 2000 simulated predictions of the PE-ratio for the evolutionary switching model (left) and the linear, single agent model (right).
Conclusion Empirical Validation

- **significant heterogeneity**, switching between two types of strategies:
  - **fundamentalists**, believing in **mean reversion**
  - **trend followers**, believing bubble to continue

- in the late nineties, the market is **dominated** by trend followers
  “dot com bubble” **triggered by fundamental news, strongly reinforced by trend following strategies**
Next lecture, afternoon

- Laboratory Experiments:
  test the behavioral assumptions in laboratory experiments with expectational feedback from human subjects

- papers can be obtained at:
  CeNDEF website http://www.fee.uva.nl/cendef